**Topic:** The Great Theorem, Evaluating the sum of square reciprocals

**Notes on Topic:**

**Great Theorem:** Evaluating

Notes: From the works left behind by Euler, it is hard to choose just one that showcases

This one was chosen for its history (rendering it important and provocative), it was one of Euler’s early triumphs solidifying his reputation for mathematical genius, and finally his ability to turn an individual solution into a string of equally impressive and unexpected ones

This particular series had been examined by the Bernoulli brothers and Leibniz, stumping them all, along with the rest of the mathematical community

They all had known that it summed to less than 2, but no one was able to find the precise solution

Euler carried out the series and calculated it up to 20 places, to try and recognize the sum, he found it was tending toward 1.6449, but his number did not look familiar to him

Finally he discovered the key to unlocking this mystery, he wrote with enthusiasm, “...quite unexpectedly I have found an elegant formula...depending on .”

Euler needed two tools to derive this formula, the sine function, and the infinitely oscillating graph is used in Euler’s argument

The elementary properties of the sine function combined with the power of calculus give us,

, this is what is meant when we hear sin x is written as an infinitely long polynomial

This was one of the clues Euler needed

The other fact came from simple algebra

Since the sine expansion suggested an endless polynomial, Euler decided to examine the properties and behavior of ordinary finite polynomials and from there take an extension to the infinite case

Suppose P(x) is an degree polynomial and as its n roots we have x=a, x=b, x=c, … , x=d. So, P(a)=P(b)=P(c)=...=P(d)=0, suppose further that P(0)=1. Then Euler knew that P(x) factors as follows,

If we examine this polynomial for x=a, x=b, … , x=d, and x=0, we see this polynomial has the properties we sought.

Euler, contemplating this equation, decided that these properties sure should hold for an infinite polynomial -- like Newton, he was a great believer in the persistence of patterns, if the pattern was valid for finite polynomials, why not extend it to infinite

The modern mathematician knows that it’s a dangerous practice to apply the finite properties to a similar infinite

This type of manipulation demands more care than Euler gave it, but perhaps Euler got lucky or had a strong mathematical intuition, in any case, his bold extension paid off

**Theorem:**

**Proof:** Euler began by introducing the function,

To Euler, f(x) is an infinite polynomial where f(0)=1, thus it can be factored, in the manner developed above, provided we can determine the roots of f. Observe for ,

by the Taylor Expansion

So long as x is not 0, amounts to solving which is the same as solving . We know this function equals 0 precisely for all .

With these considerations behind him Euler factored as follows,

Which amounts to,

(\*)

This is the key equation, Euler has related an infinite sum to an infinite product. Euler’s infinite multiplication problem would result in,

(\*\*)

Now we have two infinite series (\*) (\*\*) that equal one another, which implies that each coefficient preceding all , terms must be equal. That is,

Then, by multiplying each side by -1 and recognizing that 3!=6 that gives us,

A final cross-multiplication yielded,

**Q.E.D.**

Euler was not far off with his original estimate, 1.6449…. and Jakob Bernoulli was also correct when he stated it was certainly less than 2.

What an astounding conclusion, that the sum of this infinite series has generated an answer involving which is associated with circles and numbers like 1, 4, 9, 16 arise in conjunction with squares, an outcome linking the two could have hardly been anticipated.

Even Euler was surprised by this result, the unexpectedness combined with the cleverness of his argument makes this a great theorem of the first rank

**Additional Suggested Reading**: Epilogue, Chapter 9

**Assignment:** Homework 8 Problems, 106-108, 110